

DISJOINTNESS AND ORDER PROJECTIONS IN THE VECTOR LATTICES OF ABSTRACT URYSON OPERATORS

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ABSTRACT. Projections onto several special subsets in the Dedekind complete vector lattice of orthogonally additive, order bounded (called abstract Uryson) operators between two vector lattices E and F are considered and some new formulas are provided.

1. INTRODUCTION

The study of nonlinear maps between vector lattices is a growing area and a subject of intensive investigations [4, 5, 20], where the background has to be found in the nonlinear integral operators, see e.g. [8]. The interesting class $\mathcal{U}(E, F)$ of nonlinear, order bounded, orthogonally additive operators, the so called abstract Uryson operators from a vector lattice E into a vector lattice F was introduced and studied in 1990 by Mazón and Segura de León [13, 14], and then considered to be defined on lattice-normed spaces by Kusraev and Pliev in [11, 12, 16]. If F is Dedekind complete then $\mathcal{U}(E, F)$ turns out to be a Dedekind vector lattice and, so the band or order projections are of great interest as a tool for further investigation.

In this paper some new formulas for projections in $\mathcal{U}(E, F)$ are obtained which are of interest on their own. In a forthcoming paper these formulas play an important role in the investigation of finite elements in $\mathcal{U}(E, F)$.¹

2. PRELIMINARIES

The goal of this section is to introduce some basic definitions and facts. General information on vector lattices the reader can find in the books [2, 9, 21, 22].

Recall that an element z in a vector lattice E is said to be a *component* or a *fragment* of x if $z \perp (x - z)$. The notations $x = y \sqcup z$ and $z \sqsubseteq x$ mean that $x = y + z$ with $y \perp z$ and that z is a fragment of x , respectively. The set of all fragments of the element $x \in E$ is denoted by \mathcal{F}_x . Let be $x \in E$. A

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collection $(\rho_\xi)_{\xi \in \Xi}$ of elements in E is called a *partition of x* if $|\rho_\xi| \wedge |\rho_\eta| = 0$, whenever $\xi \neq \eta$ and $x = \sum_{\xi \in \Xi} \rho_\xi$.

Definition 2.1. Let E be a vector lattice and let X be a real vector space. An operator $T: E \rightarrow X$ is called *orthogonally additive* if $T(x + y) = T(x) + T(y)$ whenever $x, y \in E$ are disjoint elements, i.e. $|x| \wedge |y| = 0$.

It follows from the definition that $T(0) = 0$. It is immediate that the set of all orthogonally additive operators is a real vector space with respect to the natural linear operations.

So, the orthogonal additivity of an operator T will be expressed as $T(x \sqcup y) = T(x) + T(y)$.

Definition 2.2. Let E and F be vector lattices. An orthogonally additive operator $T: E \rightarrow F$ is called:

- *positive* if $Tx \geq 0$ holds in F for all $x \in E$,
- *order bounded* if T maps order bounded sets in E to order bounded sets in F .

An orthogonally additive order bounded operator $T: E \rightarrow F$ is called an *abstract Uryson operator*.

The set of all abstract Uryson operators from E to F we denote by $\mathcal{U}(E, F)$. If $F = \mathbb{R}$ then an element $f \in \mathcal{U}(E, \mathbb{R})$ is called an *abstract Uryson functional*.

A positive linear order bounded operator $A: E \rightarrow F$ defines a positive abstract Uryson operator by means of $T(x) = A(|x|)$ for each $x \in E$.

We will consider an important example. The most famous examples are the nonlinear integral Uryson operators which are well known and thoroughly studied e.g. in [8], chapt.5.

Let (A, Σ, μ) and (B, Ξ, ν) be σ -finite complete measure spaces, and let $(A \times B, \mu \times \nu)$ denote the completion of their product measure space. Let $K: A \times B \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying the following conditions²:

- (C₀) $K(s, t, 0) = 0$ for $\mu \times \nu$ -almost all $(s, t) \in A \times B$;
- (C₁) $K(\cdot, \cdot, r)$ is $\mu \times \nu$ -measurable for all $r \in \mathbb{R}$;
- (C₂) $K(s, t, \cdot)$ is continuous on \mathbb{R} for $\mu \times \nu$ -almost all $(s, t) \in A \times B$.

Denote by $L_0(A, \Sigma, \mu)$ or, shortly by $L_0(\mu)$, the ordered space of all μ -measurable and μ -almost everywhere finite functions on A with the order $f \leq g$ defined as $f(s) \leq g(s)$ μ -almost everywhere on A . Then $L_0(\mu)$ is a vector lattice. Analogously the space $L_0(B, \Xi, \nu)$, or shortly $L_0(\nu)$, is defined.

Example 2.3. Given $f \in L_0(A, \Sigma, \mu)$ the function $|K(s, \cdot, f(\cdot))|$ is μ -measurable for ν -almost all $s \in B$ and $h_f(s) := \int_A |K(s, t, f(t))| d\mu(t)$ is a well

²(C₁) and (C₂) are called the Carathéodory conditions.

defined ν -measurable function. Since the function h_f can be infinite on a set of positive measure, we define

$$\text{Dom}_A(K) := \{f \in L_0(\mu) : h_f \in L_0(\nu)\}.$$

Define an operator $T : \text{Dom}_A(K) \rightarrow L_0(\nu)$ by setting

$$(2.1) \quad (Tf)(s) := \int_A K(s, t, f(t)) \, d\mu(t) \quad \nu\text{-a.e.}$$

Let E and F be order ideals in $L_0(\mu)$ and $L_0(\nu)$, respectively and K a function satisfying the conditions (C_0) - (C_2) . Then (2.1) defines an orthogonally additive, in general, not order bounded integral operator acting from E to F if $E \subseteq \text{Dom}_A(K)$ and $T(E) \subseteq F$. The operator T is called Uryson (integral) operator.

For more examples of abstract Uryson operators see [20].

In $\mathcal{U}(E, F)$ the order is introduced as follows: $S \leq T$ whenever $T - S$ is a positive operator. Then $\mathcal{U}(E, F)$ becomes an ordered vector space. If the vector lattice F is Dedekind complete then $\mathcal{U}(E, F)$ is a Dedekind complete vector lattice and the following generalizations of the well known Riesz-Kantorovich formulas for linear regular operators hold (see [2], Theorems 1.13 and 1.16).

Theorem 2.4 ([13], Theorem 3.2.). *Let E and F be vector lattices with F Dedekind complete. Then $\mathcal{U}(E, F)$ is a Dedekind complete vector lattice. Moreover, for each $S, T \in \mathcal{U}(E, F)$ and $x \in E$ the following conditions hold*

- (1) $(T \vee S)(x) = \sup\{T(y) + S(z) : x = y \sqcup z\}.$
- (2) $(T \wedge S)(x) = \inf\{T(y) + S(z) : x = y \sqcup z\}.$
- (3) $T^+(x) = \sup\{Ty : y \sqsubseteq x\}.$
- (4) $T^-(x) = -\inf\{Ty : y \sqsubseteq x\}.$
- (5) $|T|(x) = (T^+ \vee T^-)(x) = \sup\{T(y) - T(z) : x = y \sqcup z\}$
- (6) $|T(x)| \leq |T|(x).$

3. DISJOINT VECTORS AND ORDER PROJECTIONS IN $\mathcal{U}(E, F)$

Order projections are an important tool in the study of vector lattices. There are some interesting results concerning order projections in the spaces of linear, bilinear and orthogonally additive operators in vector lattices [3, 6, 7, 15, 16, 18, 19]. The peculiarities of the elements in the Dedekind complete vector lattice of abstract Uryson operators originate further new projection formulas, which encourage, in particular, the study of finite elements ([21]) in such vector lattices. Recall first some basic notions. Each band K in a Dedekind complete vector lattice F generates an order (or band) projection $\rho_K : F \rightarrow K$ defined for $f \in F$ by $\rho_K(f) = f_1$ if $f = f_1 + f_2$ with $f_1 \in K$ and $f_2 \in K^\perp$, where f_1 and f_2 are called the projections of the element f onto the bands K and K^\perp , respectively. Denote by ρ_f the band projection onto the principle band $\{f\}^{\perp\perp}$ and by ρ_f^\perp the corresponding band projection onto

$\{f\}^\perp$ in F . It is clear that $\rho_f^{\perp\perp} = \rho_f$. The following formulas for calculation of the projections (of an element $0 \leq g \in F$) onto a band K and onto a principle band $\{f\}^{\perp\perp}$ are well known and used several times:

$$\rho_K(g) = \sup\{y \in K : 0 \leq y \leq g\} \quad \text{and} \quad \rho_f(g) = \sup_{n \in \mathbb{N}} \{g \wedge (nf)\}.$$

The set of all band projections (shortly, projectors) in F is denoted by $\mathfrak{P}(F)$. Under the order $\rho' \leq \rho''$ iff $\rho' \circ \rho'' = \rho'$ and the Boolean operations

$$\rho' \wedge \rho'' = \rho' \circ \rho'', \quad \rho' \vee \rho'' = \rho' + \rho'' - \rho' \circ \rho'', \quad \rho^* = I_F - \rho,$$

where $\rho, \rho', \rho'' \in \mathfrak{P}(F)$ and I_F is the identity operator on F , the set $\mathfrak{P}(F)$ turns out to be a Boolean algebra, i.e. a distributive complemented lattice with zero 0 and unity I_F (see [9], sect.1.3.5). It is clear that $\rho \circ \rho' = \rho' \circ \rho$ and $\rho \circ \rho = \rho$ for any $\rho, \rho' \in \mathfrak{P}(F)$. The following relations among projections onto principal bands will be frequently used later on:

$$\rho_f \wedge \rho_g = \rho_{f \wedge g}, \quad \rho_f \circ \rho_f^\perp = 0, \quad \rho_g g = g, \quad \rho_f(f \wedge g) = f \wedge g, \quad \rho_f(v) + \rho_f^\perp v = v.$$

As a representative sample we show the first of them. Take $v \in F$. Then

$$\begin{aligned} (\rho_f \wedge \rho_g)v &= (\rho_f \circ \rho_g)v = \rho_f(\rho_g v) = \rho_f\left(\sup_{n \in \mathbb{N}} \{v \wedge ng\}\right) \\ &= \sup_{m \in \mathbb{N}} \left\{ \sup_{n \in \mathbb{N}} \{v \wedge ng\} \wedge mf \right\} = \sup_m \sup_n \{v \wedge \inf\{m, n\}g \wedge f\} \\ &= \sup_{k \in \mathbb{N}} \{v \wedge k(f \wedge g)\} = \rho_{f \wedge g}(v). \end{aligned}$$

A *partition of unity* is a family of projectors $(\rho_\xi)_{\xi \in \Xi} \subset \mathfrak{P}(F)$ such that $\rho_\xi \wedge \rho_\eta = 0$ for $\xi \neq \eta$ and $\sup_{\xi \in \Xi} \rho_\xi = I_F$.

For proving the subsequent theorem we need the following auxiliary proposition, which was proven by the nonstandard methods.

Proposition 3.1 ([10], Proposition 5.2.7.2). *Let F be a Dedekind complete vector lattice with a weak order unit³ u and $(x_\lambda)_{\lambda \in \Lambda}$ be an order bounded net in F . Then the net $(x_\lambda)_{\lambda \in \Lambda}$ order converges to an element $x \in F$ if and only if for every $\varepsilon > 0$ there exists a partition of unity $(\rho_\lambda)_{\lambda \in \Lambda}$ such that*

$$\rho_\lambda |x_\beta - x| \leq \varepsilon u, \quad \beta \geq \lambda.$$

Theorem 3.2. *Let E, F be vector lattices, F be Dedekind complete and let \mathcal{A} be the set of all weak order units in F . If the operators $T, S \in \mathcal{U}_+(E, F)$ are disjoint, then for every $x \in E$, $u \in \mathcal{A}$ and $\varepsilon > 0$ there exists a partition of unity $(\pi_\alpha)_{\alpha \in \Delta}$ in $\mathfrak{P}(F)$ and a family $(x_\alpha)_{\alpha \in \Delta}$ of fragments of x , such that*

$$\pi_\alpha(Tx_\alpha + S(x - x_\alpha)) \leq \varepsilon u \quad \text{for all } \alpha \in \Delta.$$

³An element $u \in F_+$ is a *weak order unit* if $\{u\}^{\perp\perp} = F$, i.e. except 0 there are no elements in F which are disjoint to u .

Proof. Take $x \in E$. Denote the set of all pairs $\alpha = (y, z) \in \mathcal{F}_x \times \mathcal{F}_x$ of mutually disjoint fragments of x , such that $y + z = x$, by Δ . For any $\alpha = (y, x-y) \in \Delta$ put $f_\alpha = Ty + S(x-y)$. Due to formula (2) of Theorem 2.4 the disjointness of the operators S and T implies $\inf_{\alpha \in \Delta} \{f_\alpha\} = 0$. Denote by Ξ the collection of all finite subsets of Δ ordered as usual by inclusion, i.e. $\xi \leq \xi'$ iff $\xi \subset \xi'$. Introduce the set $(y_\xi)_{\xi \in \Xi}$ of all infima of finite many elements of the set $\{f_\alpha : \alpha \in \Delta\}$, i.e. if $\xi \in \Xi$ is the finite set $\xi = \{\alpha_{\xi_1}, \dots, \alpha_{\xi_n}\}$, where $\alpha_{\xi_k} \in \Delta$ for $k = 1, \dots, n$, then

$$y_\xi = \bigwedge_{i=1}^n f_{\alpha_{\xi_i}}$$

The set $(y_\xi)_{\xi \in \Xi}$ is downwards directed and $\text{o-lim}_{\xi \in \Xi} y_\xi = 0$. By Proposition 3.1, for every $\varepsilon > 0$ and $u \in \mathcal{A}$ there exists a partition of unity $(\rho_\xi)_{\xi \in \Xi}$ in $\mathfrak{P}(F)$ such that

$$\rho_\xi(y_\xi) \leq \varepsilon u \quad \text{for all } \xi \in \Xi.$$

In particular, $\rho_\xi(f_\alpha) < \varepsilon u$ if $\xi = \alpha$.

Identify now F with a vector sublattice of the Dedekind complete vector lattice $C_\infty(Q)$ of all extended real valued continuous functions on some extremally disconnected compact space Q (more exactly with its image under some vector lattice isomorphism), where the choosen weak order unit u is mapped onto the constant function $\mathbf{1}$ on Q (see [1], Theorem 3.35). Then the order projections $(\rho_\xi)_{\xi \in \Xi}$ (of the above partition of unity) are the multiplication operators in the space $C_\infty(Q)$ generated by the characteristic functions $\mathbf{1}_{Q_\xi}$, respectively, where Q_ξ for all $\xi \in \Xi$ are closed-open subsets of Q such that $Q = \bigcup_{\xi} Q_\xi$ and $Q_\xi \cap Q_{\xi'} = \emptyset$ for every $\xi, \xi' \in \Xi$, $\xi \neq \xi'$. The supremum $\sup_{\xi \in \Xi} \rho_\xi$ is the identity operator I_F .

For $\xi \in \Xi$ and $\alpha \in \Delta$ define the set

$$A_\alpha^\xi = \{t \in Q_\xi : f_\alpha(t) < f_\beta(t), \beta \in \xi, \beta \neq \alpha\}$$

and denote by $\overline{A_\alpha^\xi}$ its closure in Q_ξ and, consequently in Q . So $\overline{A_\alpha^\xi}$ are closed-open subsets of Q for every $\xi \in \Xi$ and $\alpha \in \Delta$ and, mutually disjoint if at least one index is different $\alpha \neq \alpha'$ or $\xi \neq \xi'$. Denote by ρ_α^ξ the multiplication operator generated by the characteristic function $\mathbf{1}_{\overline{A_\alpha^\xi}}$, i.e. $\rho_\alpha^\xi(f) = f \cdot \mathbf{1}_{\overline{A_\alpha^\xi}}$ for any function $f \in C_\infty(Q)$. It is clear that ρ_α^ξ is an order projection in $C_\infty(Q)$ and $\overline{A_\alpha^\xi} \subset Q_\xi$ implies $\rho_\alpha^\xi \leq \rho_\xi$. Hence $\rho_\alpha^\xi(f_\alpha) \leq \varepsilon u$ for every $\alpha \in \Delta$ and every $\xi \in \Xi$. By what has been mentioned above the order projections ρ_α^ξ are mutually disjoint, whenever $\alpha \neq \alpha'$ or $\xi \neq \xi'$. Therefore, the order projections $\pi_\alpha = \sup_{\xi \in \Xi} \rho_\alpha^\xi$ and $\pi_{\alpha'} = \sup_{\xi \in \Xi} \rho_{\alpha'}^\xi$ are mutually disjoint as well. We show that the supremum of all π_α is the identity operator. By assuming the contrary there is an order projection γ which is disjoint to each projection

π_α what causes its disjointness to each ρ_α^ξ and finally, γ is disjoint to each ρ_ξ . This contradicts the fact that $(\rho_\xi)_{\xi \in \Xi}$ is a partition of unity. Thus $(\pi_\alpha)_{\alpha \in \Delta}$ is a partition of unity and

$$\pi_\alpha(Tx_\alpha + S(x - x_\alpha)) \leq \varepsilon u \text{ for every } \alpha \in \Delta.$$

■

Similar to the linear case (see [9], sect.3.1.5) we characterize next the disjointness of two positive abstract Uryson operators.

Theorem 3.3. *Let E, F be vector lattices, with F Dedekind complete. Two operators $S, T \in \mathcal{U}_+(E, F)$ are disjoint if and only if for arbitrary $x \in E$ and $\varepsilon > 0$ there exist a partition $(\rho_\alpha)_{\alpha \in \Delta}$ of unity in $\mathfrak{P}(F)$ and a family $(x_\alpha)_{\alpha \in \Delta} \subset \mathcal{F}_x$ such that the inequalities*

$$(3.1) \quad \rho_\alpha Tx_\alpha \leq \varepsilon Tx \quad \text{and} \quad \rho_\alpha S(x - x_\alpha) \leq \varepsilon Sx$$

hold for all $\alpha \in \Delta$.

Proof. Let $S \wedge T = 0$. For an arbitrary $x \in E$ consider in F the element $u = Sx \wedge Tx + \rho_{Sx \wedge Tx}^\perp(Sx + Tx)$. We may write

$$\begin{aligned} Sx + Tx &= Sx \wedge Tx + Sx \vee Tx, \\ Sx \vee Tx &= v_1 + v_2, \end{aligned}$$

for $v_1 = \rho_{Sx \wedge Tx}(Sx \vee Tx)$, $v_2 = \rho_{Sx \wedge Tx}^\perp(Sx \vee Tx)$ and $v_1 \perp v_2$. Notice that $v_1 \in (Sx \wedge Tx)^{\perp\perp}$. If $w \in F$ is an element such that $w \perp u$ then

$$w \perp (Sx \wedge Tx) \quad \text{and} \quad w \perp \rho_{Sx \wedge Tx}^\perp(Sx + Tx).$$

The first relation implies $w \perp (Sx \wedge Tx)^{\perp\perp}$ and therefore, $w \perp v_1$. Due to $Sx \vee Tx = \rho_{Sx \wedge Tx}(Sx \vee Tx) + \rho_{Sx \wedge Tx}^\perp(Sx \vee Tx)$ the second relation implies $w \perp \rho_{Sx \wedge Tx}^\perp(Sx \vee Tx)$, i.e. $w \perp v_2$. So we have $w \perp (v_1 + v_2)$ what together with $w \perp (Sx \wedge Tx)^{\perp\perp}$ implies $w \perp (Sx + Tx)$. Hence it is shown that $Sx + Tx \in \{u\}^{\perp\perp}$ and u is a weak order unit in $\{Sx + Tx\}^{\perp\perp}$. We claim that

$$(3.2) \quad \rho_{Sx}u \leq Sx \quad \text{and} \quad \rho_{Tx}u \leq Tx.$$

For that we establish first the two auxiliary relations

$$(a) \quad \rho_{Sx} \circ \rho_{Sx \wedge Tx}^\perp Tx = 0 \quad \text{and} \quad (b) \quad \rho_{Sx} \circ \rho_{Sx \wedge Tx}^\perp(Sx + Tx) = \rho_{Sx \wedge Tx}^\perp Sx.$$

For (a) consider

$$\begin{aligned} \rho_{Sx} \circ \rho_{Sx \wedge Tx}^\perp Tx &= \rho_{Sx \wedge Tx}^\perp \circ \rho_{Sx} Tx \\ &= \rho_{Sx \wedge Tx}^\perp \circ \rho_{Sx} \circ \rho_{Tx} Tx = \rho_{Sx \wedge Tx}^\perp \circ \rho_{Sx \wedge Tx} Tx = 0. \end{aligned}$$

Then (b) follows immediately as

$$\begin{aligned} &\rho_{Sx} \circ \rho_{Sx \wedge Tx}^\perp(Sx + Tx) \\ &= \rho_{Sx} \circ \rho_{Sx \wedge Tx}^\perp Sx + \rho_{Sx} \circ \rho_{Sx \wedge Tx}^\perp Tx \\ &= \rho_{Sx \wedge Tx}^\perp \circ \rho_{Sx} Sx + \rho_{Sx} \circ \rho_{Sx \wedge Tx}^\perp Tx = \rho_{Sx \wedge Tx}^\perp Sx + \rho_{Sx} \circ \rho_{Sx \wedge Tx}^\perp Tx \\ &= \rho_{Sx \wedge Tx}^\perp Sx. \end{aligned}$$

Now (3.2) can be shown as follows:

$$\begin{aligned}
\rho_{Sx}u &= \rho_{Sx}(Sx \wedge Tx + \rho_{Sx \wedge Tx}^\perp(Sx + Tx)) \\
&= \rho_{Sx}(Sx \wedge Tx) + \rho_{Sx} \circ \rho_{Sx \wedge Tx}^\perp(Sx + Tx) \\
&= \rho_{Sx \wedge Tx}(Sx \wedge Tx) + \rho_{Sx \wedge Tx}^\perp Sx \\
&\leq \rho_{Sx \wedge Tx} Sx + \rho_{Sx \wedge Tx}^\perp Sx = Sx.
\end{aligned}$$

The same argument is valid for $\rho_{Tx}u$. By the disjointness of T and S and in view of Theorem⁴ 3.2, for any $\varepsilon > 0$, there is a partition of unity $(\rho_\alpha)_{\alpha \in \Delta}$ in $\mathfrak{P}(F)$ and a family of fragments $(x_\alpha)_{\alpha \in \Delta}$ of the element x , such that

$$\rho_\alpha(Tx_\alpha + S(x - x_\alpha)) \leq \varepsilon u \quad \text{for all } \alpha \in \Delta.$$

Consequently, for any $\alpha \in \Delta$ one has

$$\begin{aligned}
\rho_\alpha Tx_\alpha &= \rho_{Tx} \circ \rho_\alpha Tx_\alpha \leq \rho_{Tx} \varepsilon u \leq \varepsilon Tx \quad \text{and} \\
\rho_\alpha S(x - x_\alpha) &= \rho_{Sx} \circ \rho_\alpha S(x - x_\alpha) \leq \rho_{Sx} \varepsilon u \leq \varepsilon Sx.
\end{aligned}$$

Let us prove the converse assertion. Take again an arbitrary element $x \in E$. According to Theorem 2.4(2) we must prove that

$$(T \wedge S)x = \inf\{Ty + Sz : x = y \sqcup z\} = 0.$$

By the assumption, for every $\varepsilon > 0$ there is a partition of unity $(\rho_\alpha)_{\alpha \in \Delta}$ in $\mathfrak{P}(F)$ and a family $(x_\alpha)_{\alpha \in \Delta} \subset \mathcal{F}_x$ with the properties (3.1). So we have

$$\begin{aligned}
(T \wedge S)x &= \inf_{y \in \mathcal{F}_x} \{Ty + Sz : x = y \sqcup (x - y)\} \\
&\leq \inf_{\alpha \in \Delta} \{Tx_\alpha + S(x - x_\alpha)\} \\
&= \sup_{\alpha \in \Delta} \rho_\alpha \left(\inf_{\alpha \in \Delta} \{Tx_\alpha + S(x - x_\alpha)\} \right) \\
&= \sup_{\alpha \in \Delta} \rho_\alpha (Tx_\alpha + S(x - x_\alpha)) \leq \varepsilon (Tx + Sx).
\end{aligned}$$

Hence $(T \wedge S)x = 0$. ■

For each set $A \subset \mathcal{U}(E, F)$ we denote by π_A the projector in $\mathcal{U}(E, F)$ onto the band $A^{\perp\perp}$ and put $\pi_A^\perp = (\pi_A)^\perp$ (the projection onto A^\perp).

A set $A \subset \mathcal{U}_+(E, F)$ is called *increasing* or *upwards directed* if for arbitrary $S, T \in A$ there exists a $V \in A$ such that $S, T \leq V$. The next formulas show, in particular, how to calculate the projection onto a band which is generated by an increasing set.

⁴A weak order unit we need for applying this theorem is $u + v$, where u , as was already mentioned, is a weak order unit in $\{Sx + Tx\}^{\perp\perp}$ and v is some weak order unit in $\{Sx + Tx\}^\perp$.

Theorem 3.4. *Let E, F be vector lattices with F Dedekind complete, $A \subset \mathcal{U}_+(E, F)$ be an increasing set. Then the following equations hold for arbitrary $T \in \mathcal{U}_+(E, F)$ and $x \in E$*

$$(3.3) \quad (\pi_A T)(x) = \sup_{\substack{\varepsilon > 0 \\ S \in A}} \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{\rho T y + \rho^\perp T x : \rho S(x - y) \leq \varepsilon S x\},$$

$$(3.4) \quad (\pi_A^\perp T)(x) = \inf_{\substack{\varepsilon > 0 \\ S \in A}} \sup_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{\rho T y : \rho S y \leq \varepsilon S x\}.$$

Proof. Both formulas are proved by a similar argument. So, only the second one will be proved. Denote for some $x \in E$ the right-hand side of (3.4) by $\vartheta(T)(x)$. It is clear that the map $\vartheta(T) : E \rightarrow F$ is order bounded and positive. If $x = x_1 + x_2$, $x_1 \perp x_2$, then every fragment of $y \in \mathcal{F}_x$ has a representation $y = y_1 + y_2$, where $y_i \in \mathcal{F}_{x_i}$ and $i = 1, 2$. Therefore $\vartheta(T)$ is an orthogonally additive operator. For $T \in \mathcal{U}_+(E, F)$ put $\kappa(T) = T - \vartheta(T)$ and prove $\kappa(T) = \pi_A T$. Since for any $y \in \mathcal{F}_x$ and $\rho \in \mathfrak{P}(F)$

$$Tx = \rho T y + \rho^\perp T y + T(x - y) \text{ implies } Tx - \rho T y = \rho^\perp T x + \rho T(x - y)$$

one has

$$\begin{aligned} \kappa(T)(x) &= Tx - \vartheta(T)(x) \\ &= \sup_{\substack{\varepsilon > 0 \\ S \in A}} \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{Tx - \rho T y : \rho S y \leq \varepsilon S x\} \\ &= \sup_{\substack{\varepsilon > 0 \\ S \in A}} \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{\rho^\perp T x + \rho T(x - y) : \rho S y \leq \varepsilon S x\}, \end{aligned}$$

what may be written as

$$\kappa(T)(x) = \sup_{\substack{\varepsilon > 0 \\ S \in A}} \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{\rho^\perp T x + \rho T y : \rho S(x - y) \leq \varepsilon S x\}.$$

The order ideal, generated by A is order dense in $A^{\perp\perp}$. Therefore a net of operators $(T_\gamma)_{\gamma \in \Gamma} \subset \mathcal{U}_+(E, F)$ exists which belongs to the ideal generated by A such that

$$T_\gamma = \sum_{i=1}^{n(\gamma)} \lambda_i S_i, \text{ where } S_i \in A, n(\gamma) \in \mathbb{N}, \gamma \in \Gamma, \lambda_i \in \mathbb{R}_+$$

and $T_\gamma \uparrow \pi_A T$ ([2], Theorems 3.3 and 3.4). Using the fact that A is an increasing set one has

$$(T_\gamma)_{\gamma \in \Gamma} \subset \bigcup_{\substack{S \in A \\ n \in \mathbb{N}}} [0, nS].$$

Fix $\gamma_0 \in \Gamma$. Then $T_{\gamma_0} \leq nS_0$ for some $S_0 \in A$ and $n \in \mathbb{N}$. For arbitrary $\varepsilon > 0$ there exist $\rho \in \mathfrak{P}(F)$ and $y \in \mathcal{F}_x$ such⁵ that $\rho S_0(x - y) \leq \varepsilon S_0 x$. Hence

⁵For example, $y = x$ and arbitrary $\rho \in \mathfrak{P}(F)$.

for those y and ρ , and due to $T_{\gamma_0} \leq \pi_A T \leq T$, one gets

$$\begin{aligned} T_{\gamma_0} x &\leq \rho T_{\gamma_0}(x - y) + \rho T y + \rho^\perp T x \\ &\leq \rho n S_0(x - y) + \rho T y + \rho^\perp T x \leq \varepsilon n S_0 x + \rho T y + \rho^\perp T x. \end{aligned}$$

So by passing first to the infimum (with respect to $y \in \mathcal{F}_x$ and $\rho \in \mathfrak{P}(F)$) and subsequently, to the supremum (with respect to $\varepsilon > 0$ and $S \in A$) on the right-hand side⁶ of the last inequalities one obtains

$$\begin{aligned} T_{\gamma_0} x &\leq \varepsilon n S_0 x + \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{ \rho T y + \rho^\perp T x : \rho S_0(x - y) \leq \varepsilon S_0 x \} \\ &\leq \varepsilon n S_0 x + \sup_{\substack{\varepsilon > 0 \\ S \in A}} \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{ \rho T y + \rho^\perp T x : \rho S(x - y) \leq \varepsilon S x \} \\ &\leq \varepsilon n S_0 x + \kappa(T)(x). \end{aligned}$$

Since ε is arbitrary thus $T_{\gamma_0} x \leq \kappa(T)(x)$ is proved. Consequently $\sup_{\gamma \in \Gamma} T_\gamma x \leq \kappa(T)(x)$ and

$$(\pi_A T)x = \sup_{\gamma \in \Gamma} T_\gamma x \leq \kappa(T)(x) \text{ implies } \vartheta(T)(x) \leq (\pi_A^\perp T)x.$$

Since $x \in E$ is arbitrary one has $\vartheta(T) \leq \pi_A^\perp T$.

The converse inequality is proved as follows. For arbitrary $T \in \mathcal{U}_+(E, F)$ the following holds

$$\vartheta(\pi_A^\perp T) \leq \vartheta(T) = \vartheta(\pi_A^\perp T) + \vartheta(\pi_A T).$$

On the other hand, by what has been proved, one has

$$\vartheta(\pi_A T) \leq \pi_A^\perp \pi_A T = 0.$$

Therefore $\vartheta(T) = \vartheta(\pi_A^\perp T)$ and $\kappa T = T - \vartheta T = \pi_A T + \pi_A^\perp T - \vartheta(\pi_A^\perp T)$. Followingly, in order to conclude $\kappa(T) = \pi_A T$ it remains to show that $\vartheta(\pi_A^\perp T) = \pi_A^\perp T$. So, let be $C = \pi_A^\perp T$ and $S \in A$. Then $C \geq 0$ and $C \wedge S = 0$. If one shows $\kappa(C) = 0$, then $\kappa(\pi_A^\perp T) = \pi_A^\perp T - \vartheta(\pi_A^\perp T)$ implies $\pi_A^\perp T = \vartheta(\pi_A^\perp T)$.

By Theorem 3.3 for any $\varepsilon > 0$ and $x \in E$ there exist a partition of unity $(\rho_\alpha)_{\alpha \in \Delta}$ in $\mathfrak{P}(F)$ and a family $(x_\alpha)_{\alpha \in \Delta} \subset \mathcal{F}_x$ such that

$$\rho_\alpha C x_\alpha \leq \varepsilon C x \text{ and } \rho_\alpha S(x - x_\alpha) \leq \varepsilon S x.$$

Then⁷ one has

$$\begin{aligned} \varepsilon C x &\geq \rho_\alpha C x_\alpha = \rho_\alpha(\rho_\alpha C x_\alpha + \rho_\alpha^\perp C x) \\ &\geq \rho_\alpha \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{ \rho C y + \rho^\perp C x : \rho S(x - y) \leq \varepsilon S x \}, \end{aligned}$$

⁶ in both cases without touching the term $\varepsilon n S_0 x$.

⁷ by using the relations $\rho \circ \rho = \rho$ and $\rho \circ \rho^\perp = 0$.

what implies

$$\begin{aligned}\varepsilon Cx &\geq \sup_{\alpha \in \Delta} \rho_\alpha \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{\rho Cy + \rho^\perp Cx : \rho S(x - y) \leq \varepsilon Sx\} \\ &= \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{\rho Cy + \rho^\perp Cx : \rho S(x - y) \leq \varepsilon Sx\}.\end{aligned}$$

Observe that the left-hand side of the inequality does not depend on S and observe that the expression on the right-hand side increases if ε decreases. For fixed $\varepsilon_0 > 0$ one has then

$$\varepsilon_0 Cx \geq \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{\rho Cy + \rho^\perp Cx : \rho S(x - y) \leq \varepsilon_0 Sx\}$$

and for $0 < \varepsilon' < \varepsilon_0$

$$\begin{aligned}\varepsilon_0 Cx \geq \varepsilon' Cx &\geq \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{\rho Cy + \rho^\perp Cx : \rho S(x - y) \leq \varepsilon' Sx\} \\ &\geq \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{\rho Cy + \rho^\perp Cx : \rho S(x - y) \leq \varepsilon_0 Sx\}.\end{aligned}$$

It follows

$$\varepsilon_0 Cx \geq \sup_{\substack{\varepsilon' > 0 \\ S \in A}} \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{\rho Cy + \rho^\perp Cx : \rho S(x - y) \leq \varepsilon' Sx\}$$

and so, $\varepsilon_0 Cx \geq \kappa(C)$ for arbitrary $\varepsilon_0 > 0$. This means $\kappa(C) = 0$. \blacksquare

For the projections onto the principal bands in $\mathcal{U}_+(E, F)$ the following formulas are obtained as special cases from (3.3) and (3.4).

Corollary 3.5. *Let E, F be the same as in the Theorem 3.4. Then for arbitrary $S, T \in \mathcal{U}_+(E, F)$ and $x \in E$ the following formulas⁸ hold*

$$\begin{aligned}(\pi_S T)x &= \sup_{\varepsilon > 0} \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{\rho Ty + \rho^\perp Tx : \rho S(x - y) \leq \varepsilon Sx\} \\ (\pi_S^\perp T)x &= \inf_{\varepsilon > 0} \sup_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{\rho Ty : \rho Sx \leq \varepsilon Sx\}.\end{aligned}$$

Another formula for $(\pi_S^\perp T)x$ is obtained as follows. First, notice that the equality $\rho_{Sx}^\perp((\pi_S^\perp T)x) = \rho_{Sx}^\perp(Tx)$ holds. Indeed, ρ_{Sx}^\perp belongs to $\mathfrak{P}(F)$ and $\rho_{Sx}^\perp S(x - 0) \leq \varepsilon Sx$ for every $\varepsilon > 0$. Therefore

$$\begin{aligned}\rho_{Sx}^\perp(\pi_S T)x &= \sup_{\varepsilon > 0} \inf_{\substack{y \in \mathcal{F}_x \\ \rho \in \mathfrak{P}(F)}} \{\rho_{Sx}^\perp(\rho Ty + \rho^\perp Tx) : \rho S(x - y) \leq \varepsilon Sx\} \\ (\text{for } y = 0 \text{ and } \rho = \rho_{Sx}^\perp) &\leq \sup_{\varepsilon > 0} \{\rho_{Sx}^\perp \circ (\rho_{Sx}^\perp(T0) + \rho_{Sx}^\perp(Tx)) : \rho_{Sx}^\perp S(x) \leq \varepsilon Sx\} \\ &= \rho_{Sx}^\perp \circ (\rho_{Sx}^\perp(T0) + \rho_{Sx}^\perp(Tx)) = \rho_{Sx}^\perp \circ \rho_{Sx}(Tx) = 0,\end{aligned}$$

⁸For $S \in \mathcal{U}_+(E, F)$ the projections onto the bands $\{S\}^{\perp\perp}$ and $\{S\}^\perp$ are denoted by π_S and π_S^\perp , respectively.

i.e. $\rho_{Sx}^\perp(\pi_S T)x = 0$. Second, for every element $\rho'(Ty)$ with $\rho' \in \mathfrak{P}(F)$ and $y \in \mathcal{F}_x$ one has

$$(\rho_{Sx} \circ \rho')Ty = (\rho_{Sx} \wedge \rho')Ty = \rho(Ty),$$

where $\rho \in [0, \rho_{Sx}]$. Then, in particular,

$$\begin{aligned} (3.5) \quad (\pi_S^\perp T)x &= (\rho_{Sx}^\perp + \rho_{Sx})(\pi_S^\perp T)x \\ &= \rho_{Sx}^\perp(Tx) + \inf_{\varepsilon > 0} \sup_{\substack{y \in \mathcal{F}_x \\ \rho \in [0, \rho_{Sx}]}} \{\rho(Ty) : \rho(Sy) \leq \varepsilon Sx\}. \end{aligned}$$

Let E, F be vector lattices. Fix $\varphi \in \mathcal{U}(E, \mathbb{R})$ and $u \in F$. The one-dimensional (rank-one) abstract Uryson operator $\varphi \otimes u : E \rightarrow F$ is defined as $(\varphi \otimes u)x = \varphi(x)u$. The projections onto the band $\{\varphi \otimes u\}^{\perp\perp}$ and onto its orthogonal complement are special cases of Corollary 3.5 and can be calculated as follows.

Proposition 3.6. *Let E, F be the same as in the Theorem 3.4. Let $\varphi \otimes u$ be a positive abstract rank-one Uryson operator, where $\varphi \in \mathcal{U}_+(E, \mathbb{R})$, $u \in F_+$ and $T \in \mathcal{U}_+(E, F)$. Then for $x \in E$ the following formulas hold*

$$(3.6) \quad (\pi_{\varphi \otimes u} T)x = \sup_{\varepsilon > 0} \inf_{y \in \mathcal{F}_x} \{\rho_u(Ty) : \varphi(x - y) \leq \varepsilon \varphi(x)\}$$

$$(3.7) \quad (\pi_{\varphi \otimes u}^\perp T)x = \rho_u^\perp(Tx) + \inf_{\varepsilon > 0} \sup_{y \in \mathcal{F}_x} \{\rho_u(Ty) : \varphi(y) \leq \varepsilon \varphi(x)\}.$$

Proof. It is sufficient to prove only formula (3.7). For $\rho \in [0, \rho_u]$ the expression on the right side of (3.5) is $\varphi(y)\rho u \leq \varepsilon \varphi(x)u$. Due to

$$\sup_{0 \leq \rho \leq \rho_u} \varphi(y)\rho u = \varphi(y)\rho_u u = \varphi(y)u$$

the last inequality is equivalent to $\varphi(y) \leq \varepsilon \varphi(x)$. Observe that for $x \in E$ with $\varphi(x) > 0$ one has $\rho_{(\varphi \otimes u)x}(y) = \sup_{n \in \mathbb{N}} \{y \wedge n\varphi(x)u\} = \rho_u(y)$ for any $y \in F$,

i.e. $\rho_{(\varphi \otimes u)x} = \rho_u$. Therefore,

$$\rho((\varphi \otimes u)y) = \varphi(y)\rho u \leq \varepsilon \varphi(x)u = \varepsilon(\varphi \otimes u)x,$$

i.e. (if for $y \in \mathcal{F}_x$ the element x is taken) $\rho((\varphi \otimes u)x) \leq \varepsilon(\varphi \otimes u)x$. Thus the supremum on the right side of the formula (3.5) is attained at $\rho = \rho_u$. Now the conclusion is exactly the formula (3.7) \blacksquare

Corollary 3.7. *Let E be a vector lattice and $T, \varphi \in \mathcal{U}(E, \mathbb{R})$. Then the following formula holds*

$$(3.8) \quad (\pi_\varphi T)x = \sup_{\varepsilon > 0} \inf_{y \in \mathcal{F}_x} \{Ty : \varphi(x - y) \leq \varepsilon \varphi(x)\}.$$

This immediately follows from (3.6) if φ is written as $\varphi \otimes 1$.

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